MATHEMATICS

A CONSEQUENCE OF MARTIN'S AXIOM

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It is known [1] that Martin's axiom – which will be stated below – in conjunction with the negation of the continuum hypothesis implies the validity of Suslin's hypothesis, i.e. every (compact) ordered space with the Suslin property is separable. It is natural to ask whether the same implication holds for a larger class of spaces. It was shown in [2] that the answer to this question is affirmative for locally compact spaces which are hereditarily Suslin and first countable.

In this paper we prove the following result: Martin's axiom and $2^\omega > \omega_1$ imply that every locally compact Suslin space of π-weight less than continuum is separable. As a corollary of this we obtain that in [2] the condition of being *hereditarily* Suslin is superfluous.

1.1. Let $(C, \leqslant)$ be a partially ordered set. Two elements $p, q \in C$ are called compatible if there is an $r \in C$ such that $r \leqslant p$ and $r \leqslant q$, and incompatible if no such r exists. A subset $A \subset C$ is called an antichain if it consists of pairwise incompatible elements. We say that $(C, \leqslant)$ satisfies the countable antichain condition (briefly, c.a.c.) if every antichain of $(C, \leqslant)$ is countable.

A subset $D \subset C$ is called dense if for any $p \in C$ there is a $q \in D$ with $q \leqslant p$.

Now let F be a family of dense subsets of C. A subset $G \subset C$ is called C-generic over F if it satisfies the following conditions

(i) $p_1, p_2 \in G \to (\exists p_3 \in G)(p_3 \leqslant p_1 \wedge p_3 \leqslant p_2)$

(ii) if $D \in F$ then $G \cap D \neq \emptyset$.

Now consider the following statement where α is a fixed infinite cardinal:

(A_α) If $(C, \leqslant)$ is a partially ordered set satisfying c.a.c. and F is a family of dense subsets of C, $|F| \leqslant \alpha$, then there exists a set $G \subset C$ which is C-generic over F.

We call Martin's axiom, (MA), the statement "for each $\alpha < 2^\omega$ (A_α) holds".

It is known (cf. [1] or [3]) that it is consistent to assume (MA) and "2^ω is an arbitrarily large regular cardinal" at the same time.

1.2. In what follows we shall investigate some consequences of $(MA) + (2^\omega > \omega_1)$, which we shall briefly denote by (MA^*). First we need a lemma which was probably first proved by F. Rowbottom. Since it has not been published yet, we give a proof of it here.

1.3. *Definition.* A space R is said to have the Suslin property if any family of pairwise disjoint open subsets of R is countable. (For this, we also say briefly that R is a Suslin space.)

1.4. *Lemma.* If (A_{ω_1}) holds then any product of spaces with the Suslin property again has the Suslin property.

Proof. We shall make use of the following fact (see e.g. [2]): If (A_{ω_1}) holds, X has the Suslin property and $\mathfrak{G}$ is a family of open subsets of X with $|\mathfrak{G}|=\omega_1$, then there is a $\mathfrak{G}' \subset \mathfrak{G}$ with $|\mathfrak{G}'|=|\mathfrak{G}|=\omega_1$ such that $\mathfrak{G}$ is centered, i.e. any finitely many members of $\mathfrak{G}'$ have a non-empty intersection.

First we shall show that, assuming (A_{ω_1}), the product $Z=X_1\times X_2$ of two Suslin spaces X_1 and X_2 is also Suslin. Suppose, on the contrary, that Z contains ω_1 pairwise disjoint open subsets $\{G_\xi=G_\xi{}^1\times G_\xi{}^2\colon \xi<\omega_1\}$, where $G_\xi{}^j$ $(j=1, 2)$ is open in X_j. According to the above result, there is a subset $a\subset\omega_1$ with $|a|=\omega_1$ such that the family

$$\{G_\xi{}^1\colon \xi\in a\}$$

is centered (here, of course, we cannot assume that $\xi, \eta\in a$, $\xi\neq\eta$ imply $G_\xi{}^1\neq G_\eta{}^1$). Similarly, we can find a subset $b\subset a$ with $|b|=\omega_1$ such that

$$\{G_\xi{}^2\colon \xi\in b\}$$

is centered. From this it follows then that any two members of $\{G_\xi\colon \xi\in b\}$ intersect, which is a contradiction. Now, by induction we can conclude that the product of any finite number of Suslin spaces is again Suslin space.

Finally, let $X=\times\{X_i\colon i\in I\}$, where the index set I is infinite, and suppose that $\{G_\xi\colon \xi<\omega_1\}$ is a system of pairwise disjoint elementary open subsets of X. For each $\xi<\omega_1$ there is a finite subset $I_\xi\subset I$ such that

$$G_\xi=\cap\,\{\pi_i^{-1}G_i^{(\xi)}\colon i\in I_\xi\},$$

where π_i denotes the i-th coordinate projection of X onto X_i and $G_i^{(\xi)}$ is open in X_i. According to a theorem of Erdös and Rado (cf. [4] or [5]) there exists a subset $a\subset\omega_1$ with $|a|=\omega_1$ such that the family $\{I_\xi\colon \xi\in a\}$ is quasi-disjoint, i.e. there is a set J for which

$$I_\xi\cap I_\eta=J \text{ if } \xi, \eta\in a,\ \xi\neq\eta$$

(here, again, we do not assume $I_\xi\neq I_\eta$ for $\xi\neq\eta\in a$.)

Now it is easily seen that the sets

$$G_{\xi, J}=\times\{G_i^{(\xi)}\colon i\in J\}\quad(\xi\in a)$$

are pairwise disjoint (and open) in the finite subproduct

$$X_J=\times\{X_i\colon i\in J\}.$$

However, this is impossible because it contradicts our above result.

Before we can turn to our main result, we need further definitions.

2.1. *Definition.* (J. de Groot and J. van der Slot, cf. e.g. [6] or [7]). A regular topological space X is called basis compact if it has an open base $\mathfrak{B}$ such that for any centered subsystem $\mathfrak{G} \subset \mathfrak{B}$ we have

$$\cap \{\bar{G}: G \in \mathfrak{G}\} \neq \emptyset.$$

Basis compact spaces form a generalization of complete metric spaces since a metrizable space is basis compact if and only if it is topologically complete. Also, all locally compact Hausdorff spaces are basis compact.

2.2. *Definition.* For any space X a collection $\mathfrak{P}$ of its open subsets is called a π-base of X if for any non-empty open set $G \subset X$ there is a $P \in \mathfrak{P}$ for which $P \subset G$. The smallest infinite cardinality of a π-base of X is called the π-weight of X.

Now, we are able to formulate our main result.

2.3. *Theorem.* Suppose that (MA^*) holds and X is a basis compact Suslin space for which $\pi(X) < 2^\omega$; then X is separable.

Proof. Let $\mathfrak{B}$ be an open base of X with the property indicated in the definition of basis compactness. Obviously, we can select a π-base $\mathfrak{P} \subset \mathfrak{B}$ such that

$$|\mathfrak{P}| = \pi(X) < 2^\omega.$$

Then we consider the space X^ω, i.e. the ω-fold topological product of X by itself. Thus we can write

$$X^\omega = \times \{X_i: i < \omega\},$$

where $X_i = X$ for each $i < \omega$. We shall denote by π_i the projection of X^ω onto X_i. It is easy to see that the elementary open sets of the form

$$(*) \qquad \begin{cases} G = \cap \{\pi_i^{-1} G_i : i \in J\}, \\ \text{where } J \subset \omega \text{ is finite and } G_i \in \mathfrak{B} \text{ for each } i \in J \end{cases}$$

constitute a base $\mathfrak{C}$ of X^ω "under which" it is basis compact. Also, by 1.4, X^ω is a Suslin space.

Now we consider the partially ordered set $(\mathfrak{C}, \subset)$ which satisfies c.a.c., since X^ω is Suslin. Then, for each $P \in \mathfrak{P}$ we define a dense subset $\mathfrak{S}_P$ of $\mathfrak{C}$ as follows:

$$\mathfrak{S}_P = \{G \in \mathfrak{C}: (\exists i < \omega)(\pi_i G = G_i = P)\}.$$

To see that $\mathfrak{S}_P$ is dense in $(\mathfrak{C}, \subset)$, consider any $G \in \mathfrak{C}$ of the form (*) and take a $j \in \omega \setminus J$. Now the set

$$G^* = G \cap \pi_j^{-1} P$$

obviously belongs to $\mathfrak{S}_P$ and is contained in G.

Using (MA^*) and $|\mathfrak{P}|<2^\omega$, we can find a set $\mathfrak{G}$ which is $\mathfrak{C}$-generic over $\{\mathfrak{S}_P : P \in \mathfrak{P}\}$. Then, according to property (i) of the generic sets, $\mathfrak{G}$ is centered, thus $\mathfrak{G} \subset \mathfrak{C}$ implies

$$\cap \{\bar{G} : G \in \mathfrak{G}\} \neq \emptyset.$$

Let $p=(p_0, p_1, \ldots)$ be an arbitrary point of this intersection. We claim that

$$S=\{p_i=\pi_i p : i<\omega\}$$

is a dense subset of X, and thus X is indeed separable.
Let H be an arbitrary non-empty open subset of X. Since X is regular we can find a $P \in \mathfrak{P}$ such that $\bar{P} \subset H$. According to property (ii) of the generic sets, there is a

$$G \in \mathfrak{G} \cap \mathfrak{S}_P.$$

Then $\pi_i G = P$ for some $i<\omega$, and since $p \in \bar{G}$, this implies

$$p_i \in \overline{\pi_i G} = \bar{P} \subset H, \text{ hence } S \cap H \neq \emptyset.$$

This, however, shows that S is dense in X, since H was arbitrary.

§ 3. Applications of the main theorem.

3.1. In order to obtain several further consequences of the preceeding result, we recall some notations. The smallest infinite cardinal of a dense subset of a space X is denoted by $d(X)$. If $p \in X$, $\chi(p, X)$ is the minimal cardinality of a basis of neighbourhoods of p in X. We put

$$\chi(X) = \sup \{\chi(p, X) : p \in X\}.$$

Finally, we shall denote by $c(X)$ the supremum of cardinalities of disjoint families of open sets in X. Thus, e.g. $c(X)=\omega$ means that X is Suslin.

3.2. *Corollary.* Suppose (MA^*) and let X be a basis compact Suslin space with a dense subset $S \subset X$ such that $|S|<2^\omega$, and $\chi(p, X)<2^\omega$ for each $p \in S$. Then X is separable, i.e. $d(X)=\omega$.

Proof. For any $p \in S$, let $\mathfrak{B}_p$ be a basis of neighbourhoods of p in X such that

$$|\mathfrak{B}_p| = \chi(p, X) < 2^\omega.$$

Then, as can be easily seen,

$$\mathfrak{P} = \cup \{\mathfrak{B}_p : p \in S\}$$

is a π-base of X. Now, since (MA) implies that 2^ω is regular (cf. [3]),

$$|\mathfrak{P}| \leqslant \sum_{p \in S} \chi(p, X) < 2^\omega,$$

Hence $\pi(X)<2^\omega$, and 2.3 can be applied.

Next we need a Lemma, which has nothing to do with (MA) and which, we think, has its own interest.

3.3. *Lemma.* Let X be a space with $\chi(X)=\alpha<d(X)$. Then X has a subspace $S\subset X$ such that

$$d(S)=\alpha^+ \text{ and } c(S)\leqslant c(X).$$

Proof. Let us first choose for any $p\in X$ a sequence of its neighbourhoods

$$\{V_p(\xi):\xi<\alpha\}$$

whose members form a neighbourhood basis of p in X. Then we define a *partial* function f from $X^2\times\alpha^2$ as follows:

$$f(p,q,\xi,\eta)=\begin{cases}\text{a member of } V_p(\xi)\cap V_q(\eta),\\ \text{if } V_p(\xi)\cap V_q(\eta)\neq\emptyset;\\ \text{not defined otherwise.}\end{cases}$$

Then for any $H\subset X$ we put

$$\sigma(H)=H\cup\{f(p,q;\xi,\eta):(p,q)\in H^2\wedge(\xi,\eta)\in\alpha^2\}.$$

We denote by σ^n the n-th iterate of σ and put

$$\Omega(H)=\bigcup_{n=1}^{\infty}\sigma^n(H).$$

It is easy to show that $|H|\leqslant\alpha$ implies $|\Omega(H)|\leqslant\alpha$.

Now we shall define S by transfinite induction. Let H_0 be an arbitrary subset of X with $|H_0|=\alpha$ and p_0 an arbitrary point of H_0. Suppose, that the sets H_ξ, and points p_ξ have already been defined for each $\xi<\eta$, where $\eta<\alpha^+$, so that

$$|H_\xi|=\alpha \text{ for each } \xi<\eta.$$

Then

$$B_\eta=\cup\{H_\xi:\xi<\eta\}$$

is also of power α. Therefore, since $d(X)>\alpha$, there is a point

$$p_\eta\in X\backslash\bar{B}_\eta.$$

Now we put

$$H_\eta=\Omega(\{p_\eta\}\cup B_\eta).$$

Obviously, $|H_\eta|=\alpha$ holds too, and thus the transfinite procedure can be carried out for every $\eta<\alpha^+$. Finally, we put

$$S=\cup\{H_\eta:\eta<\alpha^+\}.$$

By $|S|=\alpha^+$, to prove $d(S)=\alpha^+$ it suffices to show $d(S)\geqslant\alpha^+$. Indeed, if $D\subset S$, $|D|\leqslant\alpha$, then, since α^+ is regular, there is an $\eta<\alpha^+$ such that

$$D\subset B_\eta.$$

Then, however, $p_\eta\notin\bar{B}_\eta\supset\bar{D}$, which shows that D cannot be dense in S.

Next, if $p, q \in S$ and $\xi, \eta < \alpha$, we claim that

$$V_p(\xi) \cap V_q(\eta) = \emptyset \Leftrightarrow V_p(\xi) \cap V_q(\eta) \cap S = \emptyset.$$

This follows immediately from our construction, since $p, q \in S$ imply $p, q \in H_\nu$ for a $\nu < \alpha^+$, hence if $V_p(\xi) \cap V_q(\eta) \neq \emptyset$ then $V_p(\xi) \cap V_q(\eta) \cap \cap \Omega(H_\nu) \neq \emptyset$, but $\Omega(H_\nu) \subset S$, and this concludes this step.

Now it is obvious that this last property of S implies

$$c(S) \leqslant c(X),$$

and our proof is completed.

3.4. *Theorem.* Assume (MA^*) and let X be a basis compact Suslin space such that
(a) every closed subspace of X is basis compact;
(b) $\chi(X) = \alpha$ and $\alpha^+ < 2^\omega$.
Then X is separable.

Proof. If $d(X) < 2^\omega$, then this follows immediately from 3.2. Next we shall show, however, that this must be the case. Indeed, assume $d(X) \geqslant 2^\omega$. Then, applying 3.3, we can find a subspace $S \subset X$, for which

$$d(S) = \alpha^+ \text{ and } c(S) \leqslant c(X) = \omega.$$

Let us now consider the closed subspace

$$X_1 = \bar{S} \subset X.$$

Then X_1 is basis compact by (a), and since S is dense in X_1, we have

$$c(S) = c(X_1) = \omega,$$

hence X_1 is Suslin as well.

Now we show that $d(X_1) = d(S) = \alpha^+$. Again, $d(X_1) \leqslant d(S) = \alpha^+$ is obvious because S is dense in X_1, hence it suffices to show

$$d(X_1) > \alpha.$$

Suppose, on the contrary, that $D \subset X_1$, $|D| = \alpha$ and $\bar{D} = X_1$. For each $p \in D$, since $p \in \bar{S}$ and $\chi(p, X) \leqslant \alpha$, we can select a subset $A_p \subset S$ with $p \in \bar{A}_p$ and $|A_p| \leqslant \alpha$. Then

$$A = \cup \{A_p : p \in D\} \subset S \quad \text{and} \quad |A| \leqslant \alpha.$$

Furthermore, A must be dense in S, for if $q \in S$ and G is an open set containing q then $G \cap X_1 \neq \emptyset$, hence $G \cap D \neq \emptyset$, because D is dense in X_1. Furthermore, if $p \in D \cap G$, this also implies $G \cap A_p \neq \emptyset$ since $p \in \bar{A}_p$, hence $G \cap A \neq \emptyset$. This, however, shows that A is indeed dense in S, and this contradicts $d(S) = \alpha^+$.

Thus we obtain that X_1 is a basis compact Suslin space for which

$d(X_1) = \alpha^+$ and $\chi(X_1) \leqslant \chi(X) = \alpha$. However, by 3.2 this would imply $d(X_1) = = \omega < \alpha^+$, which is impossible.

Q.e.d.

Remark. We do not know whether the condition (a) is really necessary in this theorem, or that the condition $\alpha^+ < 2^\omega$ could be replaced by $\alpha < 2^\omega$.

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